SELF-INSULATION OF LOCALIZED STRUCTURES IN NONLINEAR TRANSFER PROCESSES

L. K. Martinson

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The nonlinear inertial effect in transfer theory is investigated when the localized structure is unchanged in spatial dimension in its evolution in a nonlinear medium with volume absorption.

A broad class of diffusional transfer processes (heat, conduction, diffusion, filtration, etc.) is described by quasilinear parabolic equations of the form

$$u_t = \operatorname{div}(k(u)\operatorname{grad} u) + f(u,\ Du,\ t,\ x),\ t>0,\ x\in \mathbf{R}^n,\ n=1,\ 2,\ \text{and}\ 3. \tag{1}$$

In recent years fundamental results have been obtained in the theory of nonlinear processes of diffusional type, elucidating a series of features of such processes taking account of nonlinearity. Nonlinear properties of the medium and, in particular, the dependence of the transfer coefficient k on the transfer potential qualitatively change the evolution of structures in such media in comparison with classical diffusional processes described by linear parabolic equations. These features appear in investigating the propagation of perturbations in nonlinear media, when the initial distribution $u_0(x) = u(x, 0)$ describing the initial profile of the structure in the corresponding problems is specified in the form of finite functions with a compact carrier. In this case such nonlinear effects as a finite velocity of propagation and spatial localization of the perturbation — inertial effects with no analog in linear theory — are observed [1-9].

Inertial properties in the diffusional propagation of a perturbation in nonlinear media appear most clearly in the form of the effect of self-insulation of evolving structures. In the presence of such an effect, the perturbation front remains motionless for a time interval of arbitrary length and the spatial region of the perturbation does not change over time. In other words, in such conditions of evolution of a localized structure, internal mechanisms of the nonlinear transfer process insulate it from the surrounding space, suppressing the tendency of the structure to broaden on account of the diffusional flux.

From a mathematical viewpoint, the self-insulation of structures means that in equations of the form in Eq. (1) there must exist finite solutions with a compact carrier which does not change in size over time, i.e., solutions for which $\operatorname{supp} u(x, t) = \operatorname{supp} u_0(x)\ \forall t \geqslant 0$.

As an example of the realization of such unusual conditions of the evolution of a localized structure in a nonlinear medium with absorption, consider the process described in Eq. (1) in which $k(u) = k_0 u^\sigma$ ($k_0 = \text{const} > 0$) and the junior term takes the form $f = -f_0 t^\alpha u^\nu$ ($f_0 = \text{const} > 0$).

In this case, after trivial scale changes, Eq. (1) may be written in the form

$$u_t = \operatorname{div}(u^\sigma \operatorname{grad} u) - \Pi (t+t_0)^\alpha u^\nu (x,\ t),\ \in \mathbf{R}^n \times (0,\ +\infty). \tag{2}$$

Here $\Pi = \text{const} > 0$ is the absorption coefficient, while the parameter $t_0 > 0$ (the time shift) is only introduced so as to eliminate the singularity in the junior term of the equation as $t \to 0$ when $\alpha < 0$.

The accurate particular solution of the quasilinear parabolic Eq. (2) is found in the case when $\alpha = -2$, $\nu = 1 - \sigma$ ($0 < \sigma < 1$, $0 < \nu < 1$). It is assumed that the structure has symmetry and the distribution of the transfer potential at any time depends solely on one radial spatial variable $r = |x|$, $r \in R^1_+ = \mathbf{R}^1 \cap \{r \geqslant 0\}$. Note that the physical formulation of the problem

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associated with the description of the transfer process requires that the desired nonnegative solution (in the general case, generalized [7]) satisfy the continuity conditions for the potential u and the flux $u^{\sigma}\,\mathrm{grad}\,u$.

The particular solution of Eq. (2) for this case will be sought formally as a function with separable variables

$$u(x,\ t) \equiv u(r,\ t) = v(r)\cdot T(t), \tag{3}$$

the time factor being chosen in the special form $T(t) = (t + t_0)^{-\frac{1}{\sigma}}$.

Then, substitution of Eq. (3) into Eq. (2) gives the following nonlinear equation for v(r):

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}v^{\sigma}\frac{dv}{dr}\right) + \sigma^{-1}v - \Pi v^{1-\sigma} = 0. \tag{4}$$

It may be established by direct verification that the generalized solution of this equation satisfying the condition $v \in C^1(\mathbf{R}^1_+)$, which ensures satisfaction of the above-noted potential and flux continuity equations, is the following finite function with a compact carrier:

$$v(r) = \begin{cases} A\left[1 - \left(\frac{r}{r_0}\right)^2\right]^{\frac{1}{\sigma}}, & 0 \leqslant r < r_0, \\ 0, & r \geqslant r_0, \end{cases} \tag{5}$$

where

$$A = \left[\frac{1}{2}\sigma(2 + n\sigma)\Pi\right]^{\frac{1}{\sigma}};\quad r_0 = (2 + n\sigma)\Pi^{\frac{1}{2}}. \tag{6}$$

Thus, when α = −2 and ν = 1 − σ, Eq. (2) has the following generalized solution:

$$u(r, t) = \begin{cases} A\left[\dfrac{1 - \left(\frac{r}{r_0}\right)^2}{t + t_0}\right]^{\frac{1}{\sigma}}, & r \in [0,\ r_0), \\ 0, & r \in \mathbf{R}^1_+ \setminus [0,\ r_0), \end{cases} \tag{7}$$

where A and r_0 are defined by Eq. (6).

The solution in Eq. (7) with a time-varying carrier describes the evolution of a structure localized in a spherical region of radius r_0 which is self-insulated throughout the entire process. At boundary points of the carrier of the solution $r = r_0$, where the function u appears against a zero unperturbed background, the flux is zero at any moment of time.

It is characteristic that the dimension of the localized structure r_0 at which the self-insulation effect is observed is not arbitrary, but uniquely determined by the values of the parameters in Eq. (2), in accordance with Eq. (6).

The solution in Eq. (7) shows that nonpropagation of the perturbation front ($r = r_0$) leads to an evolution of the structure of unique form, when the broadening of the structure over time that is customary for diffusional processes is not observed on account of volume absorption. In the present case this effect is due to the influence of the junior term in Eq. (2), since when Π = 0 no structure of the form in Eq. (7) exists

Note, in conclusion, that this effect of the self-insulation of localized structures whose evolution is described by equations of the form in Eq. (1) may also be observed in the case when $k = k_0 = \mathrm{const} > 0$, i.e., in a medium with a constant transfer coefficient [10, 11]. In this case, this effect is due exclusively to nonlinearity of the junior term in the equation.

A qualitatively similar effect may also be observed in a nonlinear medium with $k(u) = k_0 u^{\sigma}$, when no volume sources are present ($f \equiv 0$) or when "active" volume processes occur in it, i.e., when the junior term in Eq. (1) is nonnegative and depends on u [12, 13]. However, in contrast to the above case of a nonlinear medium with absorption ($f \leqslant 0$), when the self-insulation effect is observed for any $t \in [0, +\infty)$, in the case when $f \equiv 0$ or $f \geqslant 0$ the self-insulated structure is always metastably localized, i.e., may only exist for some finite time interval, after which perturbation front begins to move.

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